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A birational mapping with a strange attractor: post-critical set and covariant curves

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Abstract
We consider some two-dimensional birational transformations. One of them is a birational deformation of the Hénon map. For some of these birational mappings, the post-critical set (i.e. the iterates of the critical set) is infinite and we show that this gives straightforwardly the algebraic covariant curves of the transformation when they exist. These covariant curves are used to build the preserved meromorphic 2-form. One may also have an infinite post-critical set yielding a covariant curve which is not algebraic (transcendental). For two of the birational mappings considered, the post-critical set is finite and we claim that there is no algebraic covariant curve and no preserved meromorphic 2-form. For these two mappings with finite post-critical sets, attracting sets occur and we show that they pass the usual tests (Lyapunov exponents and the fractal dimension) for being strange attractors. The strange attractor of one of these two mappings is unbounded.

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1. Introduction

The study of dynamical systems uses the notion of sensitivity to initial conditions as a criterion of the chaotic behavior. A large set of the results of the theory of dynamical systems has been proven for hyperbolic systems (sometimes with the introduction of symbolic dynamics). Otherwise, the study of a chaotic mapping is performed using various phenomenological and/or probabilistic approaches. In this dominant approach of dynamical systems the focus is on the system seen as a transformation of *real* variables, the analysis being dominated besides computer experiments, by functional analysis and differential geometry. The study consists of orbits generated on computers, phase portraits, bifurcation analyses and computation of the Lyapunov exponents. If the attracting set is not a manifold, the fractal dimension is introduced. This phenomenological and/or *probabilistic viewpoint* corresponds to the mainstream approach of dynamical systems. Most of the examples studied in the literature correspond to the iteration of polynomial or rational mappings. Another drastically different approach can be introduced and corresponds to an *algebraic and topological approach* of a dynamical system. The mapping is seen as a dynamical system of *complex variables* (complex projective space) and is studied in the framework [1–3] of a cohomology of curves in complex projective spaces. In this *topological* viewpoint, one counts integers (fixed points, degrees) and deals with singularities with *blow-up of points* and *blow-down of curves* [4]. The matching of these two drastically different descriptions of discrete dynamical systems is far from being a simple question.

Consider a two-dimensional *reversible* mapping *K*:

$$K : (u, v) \longrightarrow (K_u(u, v), K_v(u, v)). \tag{1}$$

The components $K_u(u, v)$, $K_v(u, v)$ may be polynomials or rational. Even if both components are polynomials, the inverse transformation K^{-1} may have rational components.

In studying the dynamics of a mapping having rational components, one quickly encounters the fact that the mapping is ill-defined as a continuous one because of the existence of a finite set of indeterminacy points. The *indeterminacy* set $\mathcal{I}(K)$ of mapping *K* is the finite set of points for which a component of $K(u,v)$ has form $0/0$. Polynomial mappings have, of course, no indeterminacy set.

The *critical set* consists of those algebraic varieties that cancel the Jacobian $J[K](u, v)$ of the mapping *K*. Including also the algebraic varieties such that $J[K](u, v) = \infty$ introduces the *exceptional locus*. We denote both of them by $\mathcal{E}(K)$. Mappings with constant Jacobian have, of course, no critical set.

For *reversible* two-dimensional mappings, one may want to distinguish between bi-polynomial[3] transformations, such as the Hénon mapping [5], polynomial mappings that have a rational inverse, such as those studied in [6] from the point of view of bifurcations due to contact of phase curves (basin of boundaries, saddles) with the indeterminacy set and exceptional locus[4], and *birational* mappings.

For *birational* mappings *generally*, the iterates of $\mathcal{E}(K)$ are *not curves but blow-down into points*:

$$K^n(\mathcal{E}(K)) \longrightarrow (u_n, v_n), \qquad n = 1, 2, \ldots. \tag{2}$$

These points (u_n, v_n) form the *post-critical set* [10] (that we denote PC).

Knowing the full orbit (2) may not be easy. For instance, in [9] the orbits $K^n(\mathcal{E}(K))$ have simple closed expressions. The orbits $K^n(\mathcal{E}(K))$ may have algebraic expressions with exponentially growing degrees *in the parameters* [10]. For these examples and generically for a birational mapping, the PC [10] is 'long' (or infinite) which means that, as the iteration proceeds, an infinite set of new points (u_n, v_n) is obtained.

The PC orbit *may also be 'short'* (or finite) by which it is meant that, after a finite number of iterations, the point (u_n, v_n) settles in a fixed finite point, or in (∞, ∞), and does not leave it.

In the framework [1, 2] of a cohomology of curves in complex projective spaces, Diller and Favre have presented a method [1] that gives the conditions on the parameters for which the mapping gets a complexity [11] lower than that of the generic case. This method amounts to matching the iterates of $\mathcal{E}(K)$ to the points of $\mathcal{I}(K)$. The conditions $K^n(\mathcal{E}(K)) \in \mathcal{I}(K)$

[3] Their inverse is also polynomial transformations.

[4] In [6, 7], the notions of a set of non-definition, prefocal curve and focal point are used. See also [8].

(or $K^{(-n)}(\mathcal{E}(K^{-1})) \in \mathcal{I}(K^{-1})$) give the value of the parameter for which the mapping gets a lower complexity than that of the generic case [1, 2, 11]. In other words the *complexity reduction*, which breaks the analytically stable [1] character of the mapping, will correspond to situations where some points of the orbit of the exceptional locus ($K^n(\mathcal{E}(K))$) encounter the indeterminacy set $\mathcal{I}(K)$.

By 'complexity' is meant many quantities. When one considers the degree $d(n)$ of the numerators (or denominators) of the successive nth iterate by mapping K of a rational expression, the growth of this degree is (generically) exponential with n: $d(n) \sim \lambda^n$. The constant λ has been called the 'growth complexity' [11] and for CP_2, it is closely related to the *Arnold complexity* [12, 13]. Let us also recall that two universal (or 'topological') measures of the complexities were found to identify for many examples of birational transformations [14, 15], namely the previous (degree) growth complexity [11], Arnold complexity [12, 13, 15, 16] and the (exponential of the) topological entropy [14–17]. The topological entropy is related to the growth rate, for increasing n, of the number of fixed points of K^n [1, 14, 15, 18]. For birational mappings, it is given by the (roots of the) denominator of a *rational* generating function through the dynamical zeta function [19]

$$\zeta(t) = \exp\left(\sum_{n=1}^{\infty} \#\mathrm{fix}(K^n) \cdot \frac{t^n}{n}\right), \tag{3}$$

where # $\mathrm{fix}(K^n)$ denotes the number of fixed points at order n.

All the examples we have studied are *birational mappings* [14, 15, 17, 20, 21], and we encountered the apparent discrepancy for a mapping to have non-zero (degree growth [17, 11] or Arnold growth rate [14]) complexity, or topological entropy [15], while the orbits (almost) always *look like* curves having non-positive Lyapunov exponents. The regions where the chaos [22–24] (Smale's horseshoe, homoclinic tangles, etc) is 'hidden' should be concentrated in *extremely narrow regions*. Note that Bedford and Diller [25] showed, for the mapping of [15, 16], how to build the invariant measure corresponding to non-zero *positive* Lyapunov exponents, which corresponds to a *very slim Cantor set*. Note that this invariant *real measure* is drastically different from the *complex-measure meromorphic 2-form* of the mapping.

Furthermore, in a previous paper [10], we reported on two birational mappings presenting very similar characteristics as far as topological concepts are concerned. They share the same identification between the Arnold complexity growth rate and the (exponential of the) topological entropy [15]. The complexity reduction corresponds to the same algebraic numbers given by the same family of polynomials with integer coefficients. However, if we leave aside the algebraic-topologic description, then these two mappings show different behavior on other aspects. One mapping [9] preserves a meromorphic 2-form [10] in the whole parameter space, while the other [10] does not have a preserved meromorphic 2-form for generic values of the parameters. However, on some selected algebraic subvarieties of the parameter space, the second mapping has a meromorphic 2-form. In this case, we showed that the fixed points of the birational mapping K are such that $J[K^n] = 1$, where $J[K^n]$ is the Jacobian of K^n evaluated at the fixed point of K^n. For those cases where a meromorphic 2-form has not been found, the values of $J[K^n]$ for the fixed points of K^n are different from 1. We concluded that this mapping has no meromorphic 2-form, since if it had one, then this 2-form would have to accommodate all these 'non-standard' fixed points whose number is infinite.

In addition, we have considered [10] the visualization of the iterates of arbitrary initial points showing structures which, though similar, do not converge toward the *post-critical set*, that is, the iterates of the critical set. No conclusion was drawn on the nature of these structures. In this respect, one recalls the paper by Bedford and Diller [26] which discusses a criterion related to close approaches of the *post-critical set* to the indeterminacy locus.

In this paper we focus on *birational mappings*, seizing the opportunity to use, for this specific class of transformations, the concept of PCs [10], which we show to be straightforwardly related to algebraic covariant curves and preserved meromorphic 2-forms when they exist.

We first recall some previously analyzed mappings (K_1, K_2 and K_4) and one mapping K_3 taken from the literature and show how to obtain, from the *post-critical set* [10], the (algebraic) covariant curves and the preserved meromorphic 2-form. This analysis can be performed on either the forward mapping or the backward mapping. In both directions, the post-critical set is long.

A natural question that arises then is whether the post-critical set of a birational mapping can be 'short' in one direction and 'long' in the other direction. What kind of structures do we expect? A birational mapping of this kind would be a good example to study the matching between the two viewpoints (topological and probabilistic) of the description of discrete dynamical systems. Our aim is an attempt to link the short/long aspect of the post-critical set to the *forward invariant* set occurring in a polynomial mapping with a *strange attractor* [27].

Unfortunately, and to the best of our knowledge, most [28] of the *strange attractors*[5] in two-dimensional *invertible* mappings found in the literature are *polynomial* transformations. This stems from the fact that it is common to consider an attracting set as bounded (compact set). In a typical situation (neither a necessary nor a sufficient condition [30]), these structures arise when a mapping stretches and folds an open set, and maps its closure inside it. The unbounded chaotic trajectories that occur naturally in birational mappings are thought to be divergent orbits.

We want, here, to build a birational (one-parameter) deformation of polynomial mappings. The first mapping H_c we introduce is a birational deformation of the celebrated Hénon map [5]. The deformed birational mapping depends on a further parameter c which when fixed to zero gives back the original Hénon map. This continuous deformation will show how the Hénon *strange attractor* is modified. From the topological point of view, the deformed Hénon map has the same degree complexity for generic values of c, while the strange attractor changes and the *fractal dimension of the attractor varies* continuously as a function of the deformation parameter c. For this mapping, the post-critical set is 'short' in the forward direction (and 'long' in the backward direction). It has no covariant curve and no preserved meromorphic 2-form.

We introduce a second birational mapping K, which will show that boundedness is not required for the occurrence of an attracting chaotic set[6]. First, we will compute its degree-growth complexity [11, 21] and topological entropy [16] to show that the mapping is actually chaotic. The phase portraits of the mapping show an invariant structure. We will show that these structures *pass the usual tests* commonly used to characterize the *strange attractor* (positive Lyapunov exponent and fractal dimension). These calculations are carried out even if the mapping has unbounded orbits. Thanks to the simplicity of the mapping, the fixed points (computed up to $n = 15$) are real. These fixed points all lie on the structure. The post-critical set of K is also 'short' (in the forward direction).

This last mapping K falls in a family of maps of two-step recurrences of linear fractional transformation studied by Bedford and Kim [32] in terms of periodicities and degree-growth rate [11]. Periodicities in this type of recurrences have been studied in e.g. [33, 34].

[5] The literature on strange attractors is too large to be recalled here. Strange attractors are usually described in terms of periodic points and unstable manifolds, the genesis of the visible attractor being visualized as some kind of random walk on the union of all periodic points [29]. The relation between the strange attractors and other selected points of the large literature on chaos, the homoclinic and heteroclinic points is, to our knowledge, not a very clear one.

[6] Some authors mention the possibility, or the occurrence, of unbounded attracting sets [7, 31].

The paper is organized as follows. Section 2 deals with the computation of PCs [10] for some birational mappings. These mappings being previously published, the aim is to quickly show the deep relation between the post-critical set and the covariant curves of these known mappings. In section 3, we introduce a birationally deformed Hénon map. Here also, we want to benefit from the much studied bipolynomial Hénon map to establish the effect of the short post-critical set. Section 4 presents the second two-dimensional birational mapping that has also a short post-critical set and for which the Arnold complexity growth rate and the (exponential of the) topological entropy identify. In section 5, from the analytical expressions of the Jacobian at the fixed points of the mapping up to K^{11}, and the proliferation of what we call 'non-standard fixed points' ($J[K^n] \neq 1$), we conclude on the non-existence of a preserved[7] meromorphic (here rational[8]) 2-form. The phase portraits of the mapping show an attracting set; section 6 deals with an ergodic analysis. The Lyapunov exponents are computed and the dimension of the attracting set is given by both the Kaplan–York conjecture and the box-counting method.

2. The post-critical set and covariant curves

2.1. The birational mapping K_1

Consider the mapping K_1 analyzed[9] in [15, 16, 35]:

$$K_1 : (u, v) \longrightarrow (u', v') = \left(\frac{(u+1)\, v}{1 - \epsilon u}, \frac{u}{1 + u - \epsilon u} \right). \tag{4}$$

Its Jacobian reads as

$$J[K_1](u, v) = -\frac{u+1}{(1-\epsilon u)(1+u-\epsilon u)^2}. \tag{5}$$

Using the same terminology as in [1], the critical set is given by

$$\mathcal{E}(K_1) = \left\{ (u=-1);\, (u = 1/\epsilon);\, \left(u = \frac{1}{\epsilon - 1} \right) \right\}. \tag{6}$$

The post-critical set $K_1^n(\mathcal{E}(K_1))$ is given by

$$(-1, v) \longrightarrow \left(\frac{1+(-1)^n}{n-2-n\epsilon}, \frac{1-(-1)^n}{n-1-(n+1)\epsilon} \right),$$

$$(1/\epsilon, v) \longrightarrow \left(\frac{-1}{(n-1)\epsilon}, \frac{1}{1-(n-1)\epsilon} \right),$$

and the orbit $K_1^n(u = 1/(1-\epsilon))$ depends on v.

From the iterates of $(u = -1)$, one sees that an infinite number of points of the post-critical set lie on $u = 0$ or $v = 0$. The elimination of n in the iterates of $(u = 1/\epsilon)$ gives the algebraic curve $v - u + uv = 0$. Such algebraic curves are actually covariant [10] under the action of the birational transformation.

[7] We consider here the strict preservation of a 2-form ω ($K^*\omega = \omega$), and not the preservation of the 2-form *up to a factor* ($K^*\omega = \lambda\omega$).

[8] On all of projective space CP^2, meromorphic is actually rational. In C^2, meromorphic and rational are two different notions.

[9] The original mapping K_1 was written in the variables $(1/u, 1/v)$.

[10] Throughout the paper we will, by some abuse of language, say that a curve $m(u, v) = 0$ is a 'covariant curve' to underline the covariance of the (often polynomial) expression $m(u,v)$ by our birational transformations: $m(u', v') = \mathrm{cof}(u, v)m(u, v)$. In such a case, the curve is of course invariant, but we want to focus on the covariance of the expression $m(u,v)$ and its corresponding cofactor.

Denoting $(u', v') = K_1(u, v)$, one verifies that the K_1-covariant polynomial $m_1(u, v) = uv(v - u + uv)$ is actually such that

$$\frac{m_1(u', v')}{m_1(u, v)} = J[K_1](u, v), \tag{7}$$

and one immediately deduces [36] that the corresponding meromorphic 2-form

$$\frac{\mathrm{d}u \cdot \mathrm{d}v}{m_1(u, v)} = \frac{\mathrm{d}u' \cdot \mathrm{d}v'}{m_1(u', v')} \tag{8}$$

is preserved by the birational transformation K_1.

One remarks that as $n \to \infty$, the orbit of the critical set goes to $(0, 0)$ which is a fixed point of order 1 for K_1.

2.2. *The birational mapping K_2*

Now consider the birational mapping K_2 analyzed in [9] (see equation (9) in [9]), with $c = 2 - a - b$:

$$K_2 : (u, v) \longrightarrow (u', v') = \left(\frac{a\,uv + (b-1)\cdot v + cu}{(a-1)\cdot uv + bv + cu}, \frac{a\,uv + bv + (c-1)\cdot u}{(a-1)\cdot uv + b\,v + c\,u}\right). \tag{9}$$

The Jacobian is

$$J[K_2](u, v) = \frac{uv}{((a-1)\cdot uv + cu + bv)^3}, \tag{10}$$

and the exceptional locus reads as

$$\mathcal{E}(K_2) = \left\{(u = 0);\ (v = 0);\ \left(v = \frac{-cu}{(a-1)u + b}\right)\right\}. \tag{11}$$

The successive images of the critical set are (see equation (13) in [9]):

$$\begin{aligned}
&(0,\ v) \longrightarrow \left(\frac{b-1}{b}, 1\right) \longrightarrow \cdots \longrightarrow \left(\frac{n(b-1)}{nb - (n-1)}, 1\right),\\
&(u, 0) \longrightarrow \left(1, \frac{c-1}{c}\right) \longrightarrow \cdots \longrightarrow \left(1, \frac{n(c-1)}{nc - (n-1)}\right),\\
&\left(u, \frac{-cu}{(a-1)u + b}\right) \longrightarrow (\infty, \infty) \longrightarrow \cdots \longrightarrow \left(\frac{(n-1)a - (n-2)}{(n-1)(a-1)}, \frac{(n-1)a - (n-2)}{(n-1)(a-1)}\right).
\end{aligned} \tag{12}$$

From these iterates, one notes that an infinite number of points of the post-critical set lie respectively on $v = 1, u = 1$ and $u = v$. These three lines are actually covariant. Introducing the K_2-covariant polynomial $m_2(u, v) = (u-1)(v-1)(u-v)$, one deduces from the relation [9, 36] between the Jacobian of K_2 and the ratio of m_2 evaluated at (u, v) to that at (u', v') its image by K_2:

$$\frac{m_2(u', v')}{m_2(u, v)} = J[K_2](u, v), \tag{13}$$

yielding (see (8)) a meromorphic (rational) 2-form preserved by K_2.

Here also, as $n \to \infty$, the iterates (12) go to $(1, 1)$ which is a *fixed point* of order 1 for K_2.

2.3. The birational mapping K_3

We now consider another example taken from the analysis of strongly regular graphs [37]. The birational mapping reads as

$$
\begin{aligned}
&K_3 : (u, v) \longrightarrow (u', v') = \left(1 + \frac{28(v-u)N_{uv}}{uD_{uv}}, 1 + \frac{28(v-1)N_{uv}}{D_{uv}}\right),\\
&N_{uv} = 3uv + 3u + v, \qquad (14)\\
&D_{uv} = -(c^2+35)\cdot uv^2 + 2(c^2+7)\cdot uv - 28v^2 - (c^2-49)\cdot u.
\end{aligned}
$$

The post-critical set is infinite and the orbit is given in a closed form again allowing us to obtain algebraic covariant curves. To have a preserved meromorphic 2-form, a further covariant curve needs to consider the post-critical set of the *backward* mapping. The calculations are slightly more tedious, but still simple, and are detailed in appendix A. One obtains the following K_3-covariant polynomial, with a relation between the Jacobian of K_3 and the ratio of this K_3-covariant polynomial:

$$
\begin{aligned}
&m_3(u, v) = \frac{(u-1)(v-u)\cdot}{v-1}\cdot((c^2-49)v^2 - 2(c^2+49)v + c^2 - 49),\\
&\frac{m_3(u', v')}{m_3(u, v)} = \frac{m_3(K_3(u, v))}{m_3(u, v)} = J[K_3](u, v),
\end{aligned}
$$

which, again, enables us to deduce the corresponding meromorphic (rational) 2-form.

2.4. The birational mapping K_4, for $b = a$

The fourth mapping K_4 is taken from [10] (see equation (16) in [10]) and reads as (with $c = 2 - a - b$)

$$
K_4 : (u, v) \longrightarrow \left(\frac{b(v+1)u + (b-1)v}{(a-1)\cdot uv + a\cdot(u+v)}, \frac{c(u+1)v + (c-1)u}{(a-1)\cdot uv + a\cdot(u+v)}\right), \qquad (15)
$$

with the Jacobian

$$
J[K_4](u, v) = \frac{(a+b+c-1)\cdot uv}{((a-1)\cdot uv + a\cdot(u+v))^3}. \qquad (16)
$$

The exceptional varieties of the mapping are

$$
\mathcal{E}(K_4) = \{V_1, V_2, V_3\} = \left\{(u=0); \quad (v=0); \quad \left(u = \frac{-av}{a+(a-1)\cdot v}\right)\right\}. \qquad (17)
$$

For the parameters satisfying $b = a$, the iterates $K_4^n(V_1)$ are given by (see Appendix E in [10])

$$
\begin{aligned}
&K_4^n(V_1) = (u_n, v_n) \qquad \text{with} \qquad \sigma_1 = \frac{3a^2 - 4a + 2}{2(2a-1)},\\
&u_n = \frac{2(2a-1)T_n(\sigma_1) + (5a-4)a\cdot U_{n-1}(\sigma_1) - 2(2a-1)}{2(2a-1)T_n(\sigma_1) + (5a-4)a\cdot U_{n-1}(\sigma_1) + 2(2a-1)},\\
&v_{2n} = \frac{-2(2a-1)(5a-4)\cdot T_n(\sigma_1) - 3(3a-2)(a-2)a\cdot U_{n-1}(\sigma_1)}{4(2a-1)^2\cdot T_n(\sigma_1)},\\
&v_{2n-1} = \frac{2(2a-1)(a^2+2a-2)\cdot T_n(\sigma_1) - (3a-2)(a-2)^2 a\cdot U_{n-1}(\sigma_1)}{-2(2a-1)^2 a\cdot T_n(\sigma_1) + (a-2)(3a-2)(2a-1)a\cdot U_{n-1}(\sigma_1)},
\end{aligned}
$$

where T_n and U_n are Chebyshev polynomials of order n of, respectively, first and second kinds.

We have similar results for the iterates $K_4^n(V_3)$:

$$K_4^n(V_3) = (u_n, v_n),$$
$$u_n = \frac{2(2a-1)\cdot T_n(\sigma_1) + (3a-4)a \cdot U_{n-1}(\sigma_1) + 2}{2(2a-1)\cdot T_n(\sigma_1) + (3a-4)a \cdot U_{n-1}(\sigma_1) - 2},$$
$$v_{2n} = \frac{-4(2a-1)\cdot T_n(\sigma_1) - 6(a-1)a \cdot U_{n-1}(\sigma_1)}{2(2a-1)\cdot T_n(\sigma_1) + 3a^2 \cdot U_{n-1}(\sigma_1)},$$
$$v_{2n-1} = \frac{-2(2a-1)\cdot T_n(\sigma_1) - (5a-4)a \cdot U_{n-1}(\sigma_1)}{2(2a-1)a \cdot U_{n-1}(\sigma_1)}.$$

The iterates $K_4^n(V_2)$ read as (with $\sigma_2 = (3a-4)/2$)

$$K_4^n(V_2) = \left(1, \frac{2(2a-1)\cdot U_{n-1}(\sigma_2)}{2T_n(\sigma_2) - (5a-4)\cdot U_{n-1}(\sigma_2)}\right).$$

From the iterates $K_4^n(V_2)$, one sees that an infinite number of points in the post-critical set lie on the line $u = 1$ which is *thus* covariant by the birational transformation K_4. By inspection, one obtains the fact that the orbits $K_4^n(V_1)$ and $K_4^n(V_3)$ lie on $(2(2a-1)(u+v^2) + (5a-4)(1+u)v) = 0$ and are K_4-covariant, introducing the K_4-preserved polynomial of degree 3:

$$m_4(u, v) = (u-1)\cdot(2(2a-1)(u+v^2) + (5a-4)(1+u)v). \tag{18}$$

From the relation between the Jacobian of K_4 and the m_4-ratio

$$\frac{m_4(u', v')}{m_4(u, v)} = J[K_4](u, v), \tag{19}$$

one immediately sees that the (rational) 2-form $\mathrm{d}u\mathrm{d}v/m_4(u, v)$ is invariant.

Here also, we have for the three components $K_4^n\,(\mathcal{E}(K_4))$ at the limit $n \to \infty$

$$\begin{cases} u_n \longrightarrow 1, \\ v_n \longrightarrow \dfrac{5a-4 \pm \sqrt{3(a-2)(3a-2)}}{2(1-2a)}, \quad (+: a > 0, \quad -: a < 0) \end{cases} \tag{20}$$
$$\text{for} \qquad a \in]-\infty, 2/3] \cup [2, \infty[, \qquad a \neq 0, 1/2,$$

which are fixed points of order 1 for the birational mapping K_4.

2.5. *The birational mapping K_4, for generic a and b*

From the previous examples, one sees that the *post-critical set* is an infinite orbit which is given in a closed form enabling an elimination of the iteration index n, thus yielding an explicit expression for some algebraic covariant curves. The cofactors associated with these algebraic covariants are such that a relation like (7) occurs allowing a preserved meromorphic 2-form to exist. One also remarks that the accumulation of the post-critical set is a fixed point of the mapping.

We now consider the mapping K_4, but for generic values of the parameters a and b. For this case, the post-critical set for $v = 0$ (for generic values of a and b and along all $\mathcal{E}(K_4)$, see [10]) begins as

$$(u_1, v_1) = \left(\frac{b}{a}, \frac{1-a-b}{a}\right),$$
$$(u_2, v_2) = \left(\frac{(b-1)}{(a-1)}\frac{(C+b)}{(C+a)}, \frac{(1-a-b)}{(a-1)}\frac{(C-a-b)}{(C+a)}\right),$$

$$(u_3, v_3) = \left(\frac{f(a,b)}{f(b,a)}, \frac{(1-a-b)g(a,b)}{Cf(b,a)}\right),$$
$$(u_4, v_4) = \cdots$$

with

$$f(a,b) = (b-1)\cdot(C^2 - (a+3b)\cdot C + (3b^2 + a - b - 2)b),$$
$$g(a,b) = C^3 - 2(a+b-2)\cdot C^2 - 3(a+b-2)ab\cdot C - ab,$$
$$C = (a^2 + ab + b^2) - (a+b).$$

Do note that, in contrast with the situation encountered in the previous examples, the degree growth of (the numerator or denominator of) these (u_n, v_n) *in the parameters* a and b is now actually exponential, and thus one does not expect algebraic closed forms for the successive iterates (u_n, v_n). If these iterates had been of a closed form and if the elimination of index n from u_n and v_n were possible, then this would have given a *non-algebraic* covariant curve. For this case, this *transcendental* curve should simply shrink to $u - 1 = 0$ for $b = a$.

2.6. Sum up

For the previous examples of birational mappings, we have straightforwardly obtained, from the post-critical set, the algebraic covariant curves, enabling in a second step to build the meromorphic 2-forms preserved by the mappings. This has been possible for all cases where the orbits of the critical set are obtained in closed forms. We have found that this happens whenever the degree growth *in the parameters for the iterates of the critical set* is a polynomial. This concept of 'degree growth in the parameters of the PC' has already been introduced in [10], giving a strong relation between the post-critical set and meromorphic 2-forms (when they exist).

One may define the PC as 'integrable' when the degree growth *in the parameters* of the orbits (of the critical set) is a polynomial and 'non-integrable' otherwise. For the mappings K_1, K_2, K_3 and K_4 (for $b = a$), the corresponding PC is integrable and the mappings have algebraic covariant curves. For the mappings K_4 (generic a, b) and K_5 (see appendix B), the iterates of the critical set having an exponential degree growth in the parameters (i.e. the PC is 'non-integrable'), we claim that there are no algebraic covariant curves.

Note that the conditions on possible covariant curves for birational mappings have been considered in some recent works [38–40] in a pure mathematical approach. For instance, a mapping dependent on two parameters has been studied by Bedford and Kim in [40] where they produced covariant curves when the parameters of the mapping were restricted to algebraic curves of genus 0. In appendix C, we consider this mapping to show that the post-critical set has an exponential degree growth in the parameters and hence no covariant algebraic curves. When the parameters lie on the algebraic curves given in [40], the post-critical set has indeed a polynomial degree growth in the parameters. This example illustrates what we have shown for the mapping K_4, but here the subvarieties in the parameter space for which the post-critical set becomes integrable are rather involved, compared to the simple line $b = a$ for K_4.

Since the covariance of curves (if any) should be valid in both directions, the PC should be 'long' in both directions (backward and forward). Furthermore, when the 'long' PC is integrable in one direction, it should be integrable in the other direction. This is the case for these examples (and similar mappings).

The question whether a PC can be 'short' (finite) in one direction and 'long' (infinite) in the backward direction is worth considering. Can the corresponding birational mapping have algebraic covariant curves? If so, the correspondence, which we have shown in our examples, between the occurrence of algebraic covariant curves and 'long' (and integrable) PC is just fortunate. We may even imagine a 'pathological' case where both PCs are 'short'. A birational mapping of this kind is given in appendix D.

In section 1, we mentioned the *strange attractors* (in their simple definition) which usually arise in some polynomial mappings where the *indeterminacy set* is empty and the critical set and exceptional set are also empty (the Jacobian is a constant).

A natural question which then arises is: can an *algebraic and topological concept such as the post-critical set* [10] ('long' or 'short', 'integrable' or non-integrable) be related to the structures known as *strange attractors*? For this, we will recall the well-known (bi-polynomial[11]) Hénon map [5] and deform it birationally. Is the post-critical set of this deformed Hénon map long in both directions or is it 'long' in one direction and 'short' in the other direction? We introduce another simple birational mapping that happens to be a sub-family of transformations considered by Bedford and Kim in [32] and ask the same questions.

3. Birational deformation of the Hénon mapping

We take advantage of the much studied Hénon map [5], H_0, to construct a *birational mapping* with a non-empty indeterminacy set. The birational deformation, which we introduce, should depend on a further parameter in order to get back the original mapping. In order to be as close as possible to the dynamics of H_0, the birational deformation should not add other *real fixed points* of order 1 to the original Hénon map. Note however that this constraint is not mandatory.

Recall the classical Hénon mapping [5]

$$H_0 : (u, v) \longrightarrow (1 - au^2 + bv, u), \tag{21}$$

which, under the reversible transformation

$$U : (u, v) \longrightarrow \left(\frac{u}{1 + cv}, \frac{v}{1 + cu}\right), \tag{22}$$

becomes a deformed birational Hénon mapping H_c:

$$\begin{aligned} &H_c = H_0 \cdot U : \\ &(u, v) \longrightarrow \left(1 - au^2 + bv + U_1, u - c\frac{uv}{1 + cv}\right), \\ &U_1 = -c\frac{uv}{(1 + cu)(1 + cv)^2}((bv - au^2)c^2 \cdot v - (2au^2 + auv - 2bv) \cdot c - 2au + b), \end{aligned} \tag{23}$$

with inverse

$$H_c^{-1} = U^{-1} \cdot H_0^{-1}.$$

The deformed birational mapping H_c actually reduces to the classical Hénon map for $c = 0$.

There are four fixed points of order 1 for the mapping H_c:

$$u = (1 + cv) \cdot v, \qquad \text{with}$$
$$(a + c)c^2 \cdot v^4 + (2c + a)c \cdot v^3 - (c^2 - 2c - a) \cdot v^2 - (c + b - 1) \cdot v - 1 = 0.$$

For the usual values of the parameters $a = 1.4$ and $b = 0.3$, two fixed points are complex for a large interval of the parameter c.

[11] Its inverse is also a polynomial transformation.

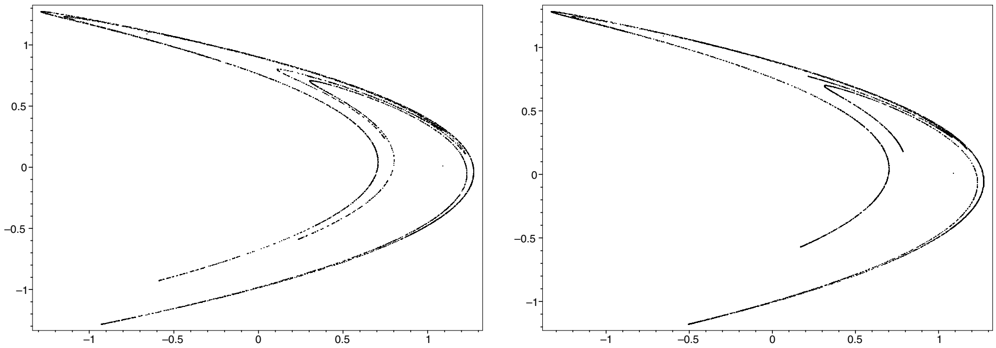

Figure 1. The attracting sets for $c = 0$ (left) and $c = 0.1$ (right)

The Jacobian of mapping H_c reads as

$$J(H_c) = -\frac{b \cdot (1 + c(u + v))}{(1 + cv)^2(1 + cu)^2}. \tag{24}$$

The indeterminacy set and exceptional locus of this birational mapping are

$$\mathcal{I}(H_c) = \{(0, -1/c), (-1/c, 0), (-1/c, -1/c), (\infty, \infty)\}, \tag{25}$$

$$\begin{aligned}\mathcal{E}(H_c) &= \{V_1, V_2, V_3\} \\ &= \{(u = -1/c), (v = -1/c), (v = -(1 + cu)/c)\}.\end{aligned} \tag{26}$$

Then the post-critical set of H_c is

$$\begin{aligned}&H_c(V_1) = (\infty, -1/c/(1 + cv)) \longrightarrow (\infty, \infty),\\ &H_c(V_2) = (\infty, \infty),\\ &H_c(V_3) = ((c^2 - a - bc)/c^2, -1/c) \longrightarrow (\infty, \infty).\end{aligned}$$

One remarks that the orbits are 'short' (ending at the fixed point (∞, ∞)), and we have only the finite point $((c^2 - a - bc)/c^2, -1/c)$ to benefit from the Diller–Favre criterion [1]. For generic values of the parameters a, b and c, the birational mapping H_c has a degree growth of $\lambda = 3$. The matching of the critical set to the indeterminacy set gives the conditions[12] on the mapping where H_c acquires less complexity than $\lambda = 3$. One finds $\lambda = 2$ for $c = 0$ corresponding to the classical Hénon map. For $a = (c - b)c$ one has $\lambda = 2.414213\ldots$ given by (the absolute value of the inverse of the smallest root of) $1 - 2t - t^2 = 0$, and for $a = (c - b + 1)c$ one obtains $\lambda = 2.618033\ldots$ given by $1 - 3t + t^2 = 0$. These three complexity reduction cases ($c = 0$, $a = (c - b)c$, $a = (c - b + 1)c$) are the only codimension-1 complexity reduction cases in the (a, b, c) parameter space. When the mapping H_c is considered along one of these cases, the Diller–Favre criterion will give the other (if any) subcases in the remaining two parameters.

It remains to see the structures that the deformed Hénon map gives. Fixing the parameters $a = 1.4$, $b = 0.3$ and for some values of the parameter c, we give in figures 1 and 2 the phase portraits of H_c. We may note in figure 2, for $c = 0.38$, the emergence of a strange attractor of

[12] These are given when the point $((c^2 - a - bc)/c^2, -1/c)$ of the post-critical set $H_c(V_3)$ encounters $(0, -1/c)$ and $(-1/c, -1/c)$ and also the condition $H_c(V_2) = (-1/c, -1/c)$.

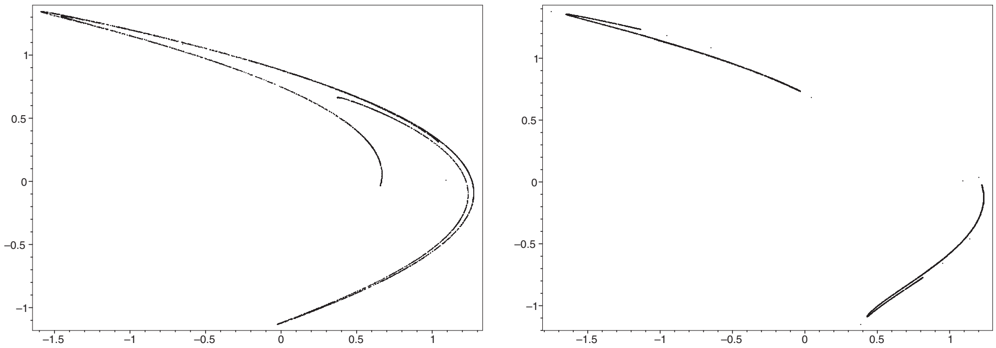

Figure 2. The attracting sets for $c = 0.3$ (left) and $c = 0.38$ (right)

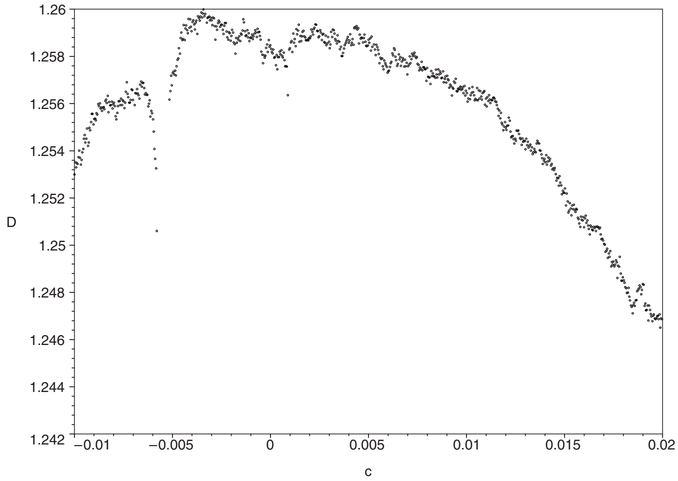

Figure 3. Dimension of the attracting set for H_c, ($a = 1.4$, $b = 0.3$) versus c.

period 2. The structures shown have a basin of attraction and are reminiscent of the original Hénon map attractor. The birational mapping H_c has a short (finite) post-critical set with only one finite point. This point is out of the basin of attraction of the strange attractor.

Figure 3 shows the *fractal dimension* (computed by using the Lyapunov exponents; see (44)–(46)) of the attracting set of the mapping H_c for some values around $c = 0$. We have considered this small interval around $c = 0$ to show how far this can be close to the fractal dimension of the original Hénon map, since the attractors (not shown) at these values of c are very similar. Note that the fractal dimensions corresponding to $c = 0.1$, $c = 0.3$ and $c = 0.38$ are respectively $D = 1.226$, $D = 1.198$ and $D = 1.137$ (for $c = 0$, the dimension is $D = 1.258$).

For the (backward) birational transformation H_c^{-1}, there are three curves in the critical set:

$$\begin{aligned}\mathcal{E}\left(H_c^{-1}\right) &= \{V_1, V_2, V_3\} \\ &= \{(v = -1/c), (cu = c - b - acv^2), (c^2vu = b + c^2v - ac^2v^3)\}.\end{aligned} \tag{27}$$

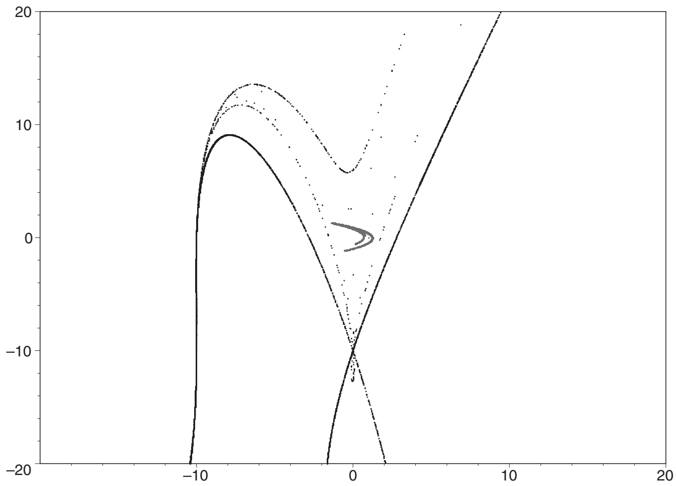

Figure 4. The attracting set of H_c^{-1} for $a = 1.4, b = 0.3, c = 0.1$, together with the strange attractor for H_c.

In contrast to the forward mapping, the orbits of the critical set for the backward H_c^{-1} are of infinite length:

$$H_c^{-1}(V_2) = (0, -1/c) \longrightarrow H_c^{-1}(V_1),$$
$$H_c^{-1}(V_3) = (\infty, \infty) \longrightarrow (-1/c, -1/c) \longrightarrow H_c^{-1}(V_1),$$
$$H_c^{-1}(V_1) = (-1/c, 0) \longrightarrow \left(0, -\frac{1+c}{bc}\right) \cdots .$$

The (parameters') degree growth in the *iterates of the critical set* is exponential $\lambda = 3$, and, thus, we cannot have algebraic expressions associated with the post-critical set. There are no algebraic covariant curves. For the values[13] $a = 1.4$, $b = 0.3$ and $c = 0.1$, the attracting set for the *backward* mapping is given in figure 4, where the *unbounded* structure is obtained for any input point. For these values of the parameters (giving a strange attractor for H_c), the backward mapping H_c^{-1} shows an unbounded attracting set in contrast to the backward Hénon map that gives divergent orbits. Figure 4 is the image of the attracting set for H_c^{-1} superposed with the attracting set for H_c for the same values of the parameters. For the mappings where the post-critical set is infinite in both directions, and when the (parameters') degree growth is exponential, one may expect (in the absence of any fractal dimension calculations) the occurrence of some non-algebraic (transcendental) covariant curves. For the mapping H_c where the post-critical set is finite in the forward direction and infinite in the backward direction, the nature of the attracting set corresponding to the backward mapping (infinite PC) is not clear.

4. A birational mapping: *K*

To build the birational mapping[14] K, we consider a combination of the Cremona inverse

$$j : \quad (x, y, z) \longrightarrow (yz, xz, xy) \tag{28}$$

[13] To obtain the strange attractor for H_c^{-1} equivalent to the attracting set for H_c, the parameters should be $(a, b) \to (a/b^2, 1/b)$.

[14] This birational mapping is a slight modification of a birational mapping considered by Bedford and Kim (equation (6.4) in [32]).

and the linear transformation

$$C : \quad (x, y, z) \longrightarrow (y + bz, z, x), \tag{29}$$

giving

$$K = C \cdot j : \quad (x, y, z) \longrightarrow ((z + by) \cdot x, xy, yz). \tag{30}$$

In the inhomogeneous variables $u = x/z$, $v = y/z$, the birational mapping becomes

$$K : \quad (u, v) \longrightarrow \left(\frac{u}{v} + bu, u\right), \tag{31}$$

with inverse

$$K^{-1} : \quad (u, v) \longrightarrow \left(v, -\frac{v}{bv - u}\right), \tag{32}$$

the linear transformation (29) becoming the collineation $(u, v) \rightarrow ((b + v)/u, 1/u)$.

The Jacobians are variables dependent and read as

$$J[K] = \frac{u}{v^2}, \qquad J[K^{-1}] = \frac{v}{(bv - u)^2}. \tag{33}$$

The indeterminacy set and the exceptional locus, for the birational mapping K, are

$$\mathcal{I}(K) = \left\{(0, 0); \left(\infty, -\frac{1}{b}\right)\right\}, \tag{34}$$

$$\mathcal{E}(K) = \{(u = 0); (v = 0)\}, \tag{35}$$

and for mapping K^{-1} they read as

$$\mathcal{I}(K^{-1}) = \{(0, 0); (\infty, \infty)\}, \tag{36}$$

$$\mathcal{E}(K^{-1}) = \{(u = bv); (v = 0)\}. \tag{37}$$

The post-critical set is the image by K of the exceptional set:

$$K(u = 0) \longrightarrow (0, 0) \longrightarrow (\infty, 0) \longrightarrow (\infty, \infty),$$
$$K(v = 0) \longrightarrow (\infty, u) \longrightarrow (\infty, \infty).$$

Here also, the orbit of the critical set $K^n(\mathcal{E}(K))$ is 'short'. By the Diller–Favre criterion [1], only $b = 0$ yields a complexity reduction, otherwise the mapping is 'analytically stable' (terminology introduced in [1]). At the value $b = 0$, the whole plane becomes a fixed point of order 6 which is easily seen from transformations (28) and (29) which are, respectively, of orders 2 and 3. The fixed points of order one $(1, 1)$, order two $(-1/2 \mp i\sqrt{3}/2, -1/2 \pm i\sqrt{3}/2)$ and order three $(1, -1), (-1, -1), (-1, 1)$ still exist, but any deviation from these exact values throws the trajectory at the fixed point of order 6.

To calculate the topological entropy [15] for the birational mapping (31), one counts the number of primitive cycles of order n, for the generic case $b \neq 0$. They are

$$\#\text{fix}(K^n) = [1, 1, 1, 0, 1, 0, 1, 1, 1, 1, 2, 2, 3, 3, 4, 5, \ldots], \tag{38}$$

from which we infer the (rational) dynamical zeta function $\zeta(t)$:

$$\zeta(t) = \frac{1}{(1 - t)(1 - t^2 - t^3)}. \tag{39}$$

The absolute value of the inverse of the smallest root of $1 - t^2 - t^3 = 0$ gives the (exponential of the) topological entropy $h = 1.324717\ldots$. This algebraic number is the *smallest Pisot*[15] *number* [42–44].

When one considers mapping (31) (in the homogeneous coordinates), the growth rate of the degrees of x (or y or z) is given by the generating function

$$g(t) = \frac{(2 + 2t + t^2) \cdot t}{1 - t^2 - t^3}, \tag{40}$$

and the degree-growth complexity (absolute value of the inverse of the smallest root of the denominator) again gives the smallest Pisot number $\lambda = 1.324717\ldots$. This degree-growth rate has been proven in [32] (and as a limiting case in [45]).

We thus see, for this mapping (and similar to the results obtained for other transformations previously studied [9, 10, 14, 16]), an identification between the growth rate of the number of fixed points of K^n and the growth rate of the degree [11, 21] of the iteration.

Following this criterion ($\lambda > 1$, $h > 1$), the birational mapping K is chaotic. However, this criterion does not properly describe the dynamics of the mapping seen as a transformation in the *real plane*. It is based on the counting of degrees or fixed points irrespective of their stability.

The fixed point of order 1 is real for any real value of b. For $b = 1$, it identifies with the fixed point at infinity. For the mapping K, the fixed point of order 1 is an unstable spiraling point for $b < 0$, a stable spiraling for $0 < b < 3/4$, a stable node for $3/4 < b < 1$, a saddle for $1 < b < 3$ and an unstable node for $b > 3$. The fixed point of order 2 is real and saddle for $b < -1$ and for $b > 3$. It *fuses* with the fixed point of order 1 at $b = 3$ and with the 'infinity' fixed point of order 1 at $b = -1$. The fixed point of order 3 is real and saddle for any real b.

Note that, similar to the mapping H_c^{-1}, the backward birational mapping K^{-1} has a 'long' post-critical set. The iterates of the critical set also have an *exponential degree growth in the parameter b*, ruling out the existence of algebraic covariant curves.

5. The birational mapping *K*: phase portraits

The phase portraits of the birational mapping K show that for $|b| > 1$, the iterates are attracted to the fixed point at infinity. For $0 < b < 1$, the fixed point of K is an attractor. For $-1 < b < 0$, the mapping is an attracting set. We show in figure 5 the attractor obtained for the value $b = -3/5$. This structure is independent of the initial starting points of the iteration and looks very much like a set of curves, a foliation of the (u, v)-plane. The structure shown in figure 5 is obtained from one starting point.

This accumulation of curves has unbounded branches. A way to 'see the global picture' amounts to performing the plot in the variables [10] $\theta_u = \arctan(u)$ and $\theta_v = \arctan(v)$. This way, the real plane is compactified to the box $[-\pi/2, \pi/2] \times [-\pi/2, \pi/2]$. Figure 6 shows the phase portrait in the variables (θ_u, θ_v). The unbounded branches of figure 5 appear in figure 6 at the bunches ($\theta_u = \pm\pi/2, \theta_v = \mp\pi/2$) and ($\theta_u = \pm\pi/2, \theta_v = \theta_{\tilde{v}}$), the larger interval for $\tilde{v}$ being $[b - 1, b]$. The attractor is thus not confined. This is consistent with the fact that it is obtained for any initial point, the basin of attraction being the whole plane.

Since the birational mapping K is not integrable according to the criterion $\lambda > 1$ or $h > 1$, one may ask whether the structures, shown in figures 5 and 6, are compatible with a preserved meromorphic 2-form or simply with covariant curves. In fact the post-critical set is

[15] For the occurrence of Pisot (and Salem) numbers for degree-growth complexity [11, 21] and for birational transformations of two variables, see [41].

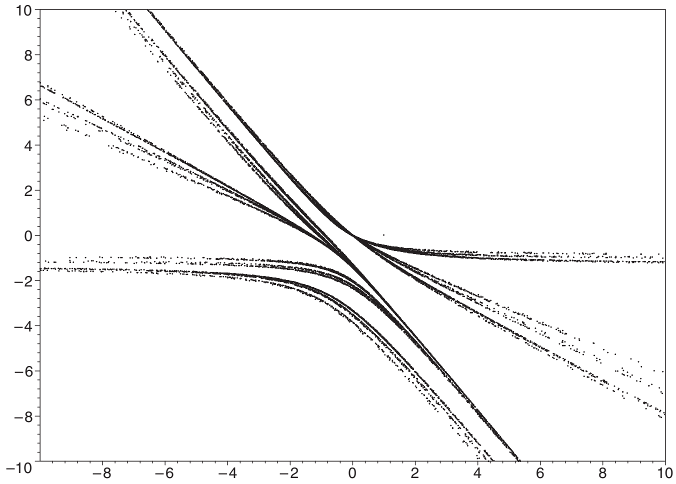

Figure 5. Phase portrait in the variables (u, v) for $b = -3/5$.

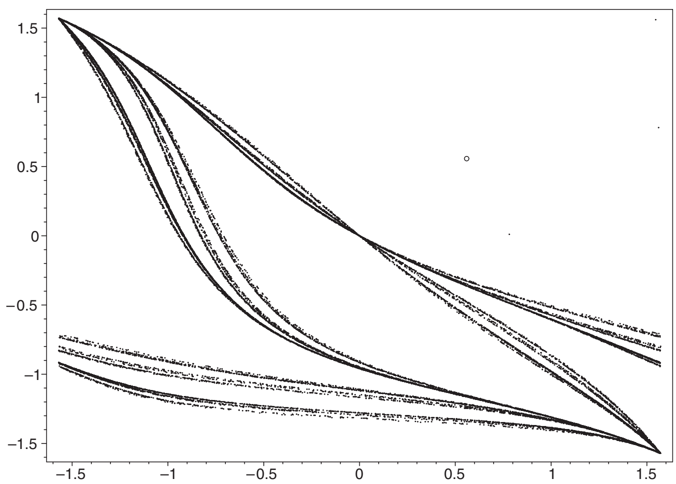

Figure 6. Phase portrait in the variables (θ_u, θ_v) for $b = -3/5$. The open circle shows the fixed point of order 1.

'short', and there is no algebraic covariant curve. In the following, we show another way to be convinced on this non-occurrence.

5.1. Non-standard fixed points

Denoting by $(u^{(n)}, v^{(n)})$, the image of a point (u, v) by transformation K^n, the preservation of a two-form $m(u, v)$ means:

$$\frac{\mathrm{d}u \cdot \mathrm{d}v}{m(u, v)} = \frac{\mathrm{d}u^{(n)} \cdot \mathrm{d}v^{(n)}}{m(u^{(n)}, v^{(n)})}. \tag{41}$$

If $J[K^n](u, v)$ denotes the Jacobian of K^n, one has

$$J[K^n](u, v) = \frac{m(u^{(n)}, v^{(n)})}{m(u, v)} = \frac{m(K^n(u, v))}{m(u, v)}. \tag{42}$$

When evaluated at the fixed point (u_f, v_f) of K^n, the Jacobian of K^n is thus equal to +1. This is what was obtained for many birational transformations we have studied [9, 15, 16]. For some of these mappings, we evaluated precisely a large number of n-cycles in order to get the dynamical zeta function [15, 16]. For all these mappings, a preserved meromorphic 2-form exists. However, we claimed for the mapping given in [10] the non-existence of a preserved meromorphic 2-form since the Jacobians evaluated at (a growing number of) the fixed points of K^n are no longer equal to 1. This mapping preserves a meromorphic 2-form in some subspaces of the parameters, and we showed, in this case, that the equality $J[K^n](u_f, v_f) = 1$ always holds, with the exception of a *finite number* of fixed points. Thus, even when a meromorphic 2-form is preserved, the equality $J[K^n](u_f, v_f) = 1$ *may be ruled out* for the fixed points of K^n that *correspond to divisors of the 2-form*. Such 'non-standard' fixed points of K^n are such that $m(u_f, v_f) = 0$ (or $m(u_f, v_f) = \infty$).

The number of these 'non-standard' fixed points [10] of K^n is an indication of the degree of $m(u,v)$, if the latter exists. When the number of such non-standard fixed points becomes very large (infinite), the existence of this algebraic 2-form may be ruled out.

Thanks to the simplicity of the mappings of this paper, the expressions of these Jacobians evaluated at the fixed points may be obtained up to a large order. For instance, at respectively the order $n = 1$, $n = 3$, $n = 10$ and $n = 11$, they read as ((u_f, v_f) are the primitive fixed points of K^n)

$$\begin{aligned}
&J[K](u_f, v_f) = 1 - b, \qquad J[K^3](u_f, v_f) = 1 + b + b^2,\\
&J[K^{10}](u_f, v_f) = \frac{(1 - b^{10})b^{10}}{(1 + b) \cdot (1 - b^5) \cdot P_1^{(10)}},\\
&J[K^{11}](u_f, v_f): \quad P_2^{(11)} \cdot J^2 + P_1^{(11)} \cdot J + P_0^{(11)} = 0,
\end{aligned} \tag{43}$$

with

$$\begin{aligned}
P_1^{(10)} &= b^8 - 4b^7 + 9b^6 - 15b^5 + 16b^4 - 14b^3 + 8b^2 - 3b + 1,\\
P_2^{(11)} &= (-2 + 8b - 22b^2 + 46b^3 - 72b^4 + 89b^5 - 87b^6 + 68b^7\\
&\quad - 41b^8 + 19b^9 - 6b^{10} + b^{11}) \cdot b,\\
P_1^{(11)} &= 1 - b + 5b^2 - 2b^3 + 8b^4 - 8b^5 + 14b^6 - 25b^7 + 16b^8\\
&\quad - 35b^9 + 13b^{10} - 38b^{11} + 21b^{12} - 34b^{13} + 22b^{14} - 19b^{15}\\
&\quad + 10b^{16} - 5b^{17} + 2b^{18} - b^{19},\\
P_0^{(11)} &= (1 - b^{11})b^{15}/(1 - b).
\end{aligned}$$

As long as the fixed points up to order 11 are sufficient to make a conclusion, no value of the parameter b gives $J[K^n]$ equal to unity for these fixed points. Considering only these fixed points, a preserved meromorphic 2-form for the birational mapping K should be, at least, of degree 134. In fact the proliferation of these non-standard fixed points gives a firm indication of the *non-existence of a preserved meromorphic 2-form*.

6. The birational mapping *K*: ergodic (probabilistic) analysis

While the 'complexity spectrum' of the mapping in [10] is involved (see figure 1 in [10]), the mapping of this paper presents the same degree-growth complexity or (exponential of the)

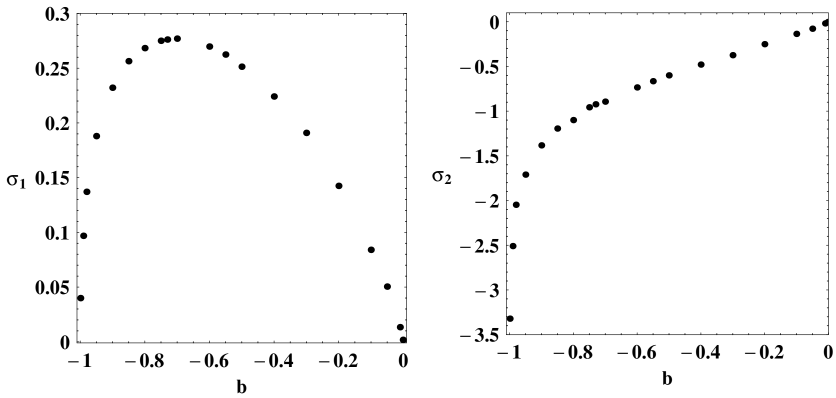

Figure 7. Positive and negative Lyapunov exponents versus the parameter b

topological entropy ($\lambda = h = 1.324717\ldots$) for any value of the parameter $b \neq 0$. Numerical investigation shows that the fixed points, up to order $n = 15$, are real for real values of the parameter b. We expect then to provide a clearer picture on the ergodic aspect of the analysis. We have seen [10] that the existence, or non-existence, of preserved meromorphic 2-forms has (paradoxically)[16] no impact on the topological complexity but drastic consequences on the numerical computation of the Lyapunov exponents.

If we believe the analysis of [10], the *Lyapunov exponents* should be zero in the case of a preserved meromorphic 2-form. This is then another check on whether the structure shown in figure 6 corresponds to a preserved meromorphic 2-form. We have computed the Lyapunov exponents for the mapping K for the large values of n as

$$\sigma_1 = \frac{1}{n}\ln(|\lambda_1|), \qquad \sigma_2 = \frac{1}{n}\ln(|\lambda_2|), \tag{44}$$

where $\lambda_{1,2}$ are the eigenvalues of

$$M^{(n)} = M(u^{(n-1)}, v^{(n-1)}) \cdots M(u^{(1)}, v^{(1)}) \cdot M(u, v), \tag{45}$$

M being the tangent map evaluated at each point $(u^{(n)}, v^{(n)}) = K^n(u, v)$.

The Lyapunov exponents for $b \in]-1, 0[$ are shown in figure 7. The largest Lyapunov exponent being *positive*, the attractor is chaotic.

Kaplan and Yorke [46] have conjectured that the dimension of an attractor can be approximated from the spectrum of Lyapunov exponents. Essentially, this conjecture corresponds to plotting the sum of Lyapunov exponents versus n (the number of Lyapunov exponents, i.e. the dimension of the system), the dimension being established by finding where the curve intercepts the n-axis by linear interpolation[17]. For a two-dimensional mapping, this gives

$$D_{\mathrm{KY}} = 1 - \frac{\sigma_1}{\sigma_2}, \tag{46}$$

where σ_1 and σ_2 are, respectively, the positive and negative Lyapunov exponents. This dimension is expected to be close to other dimensions such as box counting, information and

[16] We have made in [10] a comparison between two sets of birational transformations exhibiting totally similar results as far as *topological complexity* is concerned (degree-growth complexity, Arnold complexity and topological entropy), but *drastically different numerical* results as far as the computation of Lyapunov exponents is concerned.

[17] Note the comparison made in [47] for the correlation and Lyapunov dimensions using, for the latter, a polynomial interpolation instead of a linear one.

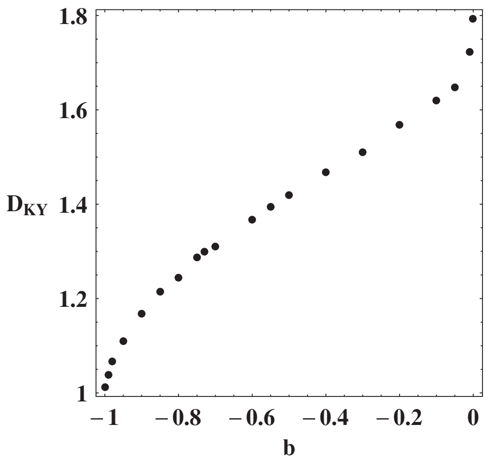

Figure 8. Kaplan-Yorke (or Lyapunov) dimension versus the parameter b

Table 1. Kaplan–York and box-counting dimensions for some values of the parameter b.

b	−0.9	−0.8	−0.6	−0.5	−0.4	−0.3	−0.2
D_{KY}	1.17	1.24	1.37	1.42	1.47	1.51	1.57
D_{box}	1.34	1.36	1.44	1.52	1.52	1.56	1.66

correlation dimension [48] for typical strange attractors. The Kaplan–Yorke dimension (also called the Lyapunov dimension) has been shown to identify with the information dimension for Baker's transformation. It has been tested for the Hénon mapping [49].

Using the Kaplan–Yorke conjecture, we can compute the (fractal) dimension of the attractor which is shown in figure 8. For $b \in]-1, 0[$, the dimension of the attractor corresponds to fractals. The attractor is thus a *strange attractor*.

The dimension obtained from the Lyapunov exponents is given from a conjecture. In order to be more convinced of the fractal nature of the attractor, we have calculated the (fractal) dimension by the box-counting method for some values of b. The box-counting dimension is believed to coincide, for most systems, with the Hausdorf–Besicovitch dimension. The box-counting dimension amounts to considering boxes of side ϵ covering the attractor and then counting the number of boxes $N(\epsilon)$ necessary to contain all the points. The box-counting dimension is the limit $\epsilon \to 0$ of

$$D_{\mathrm{box}} = -\frac{\ln(N(\epsilon))}{\ln(\epsilon)}. \tag{47}$$

The calculations are carried out in the variables θ_u, θ_v (which are in one-to-one correspondence with u, v). The results given in table 1 show an agreement with the dimension computed from the Lyapunov exponents as far as the fractal nature is concerned. Note that for this mapping, the Lyapunov (Kaplan–Yorke) dimension is less than the box-counting dimension (and is a lower bound [46]).

Remark. The simplicity of the birational mapping K makes it a good working example of many tests. For instance, it will be straightforward to compute the Lyapunov exponents and the fractal dimension from the knowledge of the first fixed points [50]. How many fixed points will be needed for that purpose? Are the fixed points sufficient to completely characterize the mapping in some ergodic analysis?

7. Conclusion

In this paper we have, first, considered four birational mappings. Three of them (K_1, K_2, K_4) have been already analyzed in previous papers and the fourth mapping (K_3) is taken from the literature on strongly regular graphs [37]. For these mappings, we have considered their post-critical set versus the occurrence of covariant curves and preserved meromorphic 2-form. In these working examples, the post-critical set is 'long' in both directions (forward and backward). We have shown that the knowledge of the orbits of the critical set allows us to obtain the algebraic covariant curves.

We have built a *birational deformation* of the Hénon map, H_c, having a 'short' post-critical set. This birational mapping shows a continuous deformation of the original Hénon map. For the values of the parameters (giving a strange attractor for H_c), the backward mapping H_c^{-1} shows an unbounded attracting set in contrast to the backward Hénon map that gives divergent orbits.

We have focused on a birational mapping K (slight modification of a birational mapping of Bedford and Kim [32]) which also has a 'short' post-critical set, with probably no preserved meromorphic 2-forms (in view of the non-standard fixed points). The mapping shows an attracting set, which *passes the tests* of being a strange attractor.

In view of these examples, we saw that birational mappings with a 'short' post-critical set had no covariant curves. We also saw that strange attractors may occur for birational mappings with a 'short' post-critical set and they are not necessarily confined.

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Appendix A. The birational mapping K_3

We consider a 3×3 matrix, acting on the three homogeneous variables (x, y, z), taken from the analysis of the matrix of strongly regular graphs [37]

$$M = \begin{bmatrix} 2 & a & b \\ 2 & -1+c & -1-c \\ 2 & -1-d & -1+d \end{bmatrix} \tag{A.1}$$

and the associated collineation C. Denoting the Cremona inverse (28) by j, the mapping $K_3 = C^{-1} \cdot j \cdot C \cdot j$ depends on four parameters. Here we fix $a = 6, b = 6, d = c$. The birational mapping in the variables $u = x/z$, $v = y/z$ is given by (14). The Jacobian[18] reads as

$$J[K_3](u, v) = -5488 \frac{vN_{uv}}{u^2 D_{uv}^3} \cdot ((c+1)uv - (c-1)u - 2v)\,((c-1) \cdot uv - (c+1)u + 2v). \tag{A.2}$$

[18] Note that the mapping depends on c^2. However, we do not rename the parameter c^2 for easy presentation; see the factorized expression (A.2) in the Jacobian.

The exceptional varieties of the mapping are

$$\mathcal{E}(K_3) = \{V_1, V_2, V_3, V_4, V_5, V_6\}$$
$$= \left\{ \begin{matrix} (u=0);\ \left(v = \frac{-3u}{1+3u}\right);\ (v=0);\ \left(v = \frac{u(c+1)}{cu-u+2}\right); \\ \left(v = \frac{u(c-1)}{cu+u-2}\right);\ (D_{uv}=0) \end{matrix} \right\}.$$

The post-critical set for V_1 is variable dependent. The orbit for V_3 reads as

$$K_3^n(V_3) = \left(\frac{1}{2}(1+(-1)^n) + \frac{1}{2}(1-(-1)^n)v_n,\, v_n\right),$$
$$v_n = \frac{f(c)-f(-c)}{g(c)-g(-c)},$$
$$f(c) = (c-7)\cdot(c-21-(-1)^n(3c-7))\cdot(c+1)^n,$$
$$g(c) = (c-7)\cdot(c+21-(-1)^n(3c+7))\cdot(c-1)^n.$$

The orbit for V_2, after $(1, 1) \rightarrow (\infty, \infty)$, gives a similar expression to V_3 and reads as $(n \geqslant 3)$

$$K_3^n(V_2) = \left(\frac{1}{2}(1-(-1)^n) + \frac{1}{2}(1+(-1)^n)v_n,\, v_n\right),$$
$$v_n = \frac{f(c)+f(-c)}{g(c)+g(-c)},$$
$$f(c) = (c-7)\cdot(3c+71+(-1)^n(c+21))\cdot(c+1)^{n-2},$$
$$g(c) = (c-7)\cdot(3c-7+(-1)^n(c-21))\cdot(c-1)^{n-2}.$$

The orbit for V_5 is identical, with the change $c \rightarrow -c$, to the orbit for V_4 which reads as

$$K_3^n(V_4) = \left(\frac{(c-1)(c+7)^n-(c+1)(c-7)^n}{f(c)\cdot v_n+(-1)^n f(-c)}\cdot v_n,\, v_n\right),$$
$$v_n = \frac{1}{2}(1-(-1)^n)\cdot\frac{c+7}{c-7} + \frac{1}{2}(1+(-1)^n)\cdot\frac{c-7}{c+7},$$
$$f(c) = (c-1)\cdot(c+(-1)^n 7)\cdot(c+7)^{n-1}.$$

Finally, the post-critical set $K_3^n(V_6)$ (which, after (∞, ∞), is identical to $K_3^n(V_3)$ with $c \rightarrow -c$ and a shift in n) reads:

$$K_3^n(V_6) = \left(\frac{1}{2}(1+(-1)^n) + \frac{1}{2}(1-(-1)^n)\cdot v_n,\, v_n\right),$$
$$v_n = \frac{f(c)-f(-c)}{f(c)g(c)-f(-c)g(-c)},$$
$$f(c) = (c-7)\cdot(3c+7+(c+21)(-1)^n)\cdot(c+1)^{n+1},$$
$$g(c) = \frac{1}{2}(1-(-1)^n)\cdot\frac{c+7}{c-7} + \frac{1}{2}(1+(-1)^n)\cdot\frac{c-7}{c+7}.$$

From these orbits, one easily finds the covariant curves. The post-critical set of $K_3^n(V_3)$ gives $u = 1$ for n even and $v = u$ for n odd; the covariant curve is thus $(u-1)(v-u) = 0$. The post-critical set of $K_3^n(V_4)$ gives $v-(c-7)/(c+7) = 0$ for n even and $v-(c+7)/(c-7) = 0$ for n odd, leading to the covariant curve $C_2 = ((c+7)\cdot v-c+7)((c-7)\cdot v-c-7) = 0$. The post-critical set of $K_3^n(V_6)$ gives the same covariant curve as $K_3^n(V_3)$.

These curves are covariant but, alone, they do not construct a preserved meromorphic 2-form (see (8)). One obtains

$$m(u, v) = (u-1)(v-u)\cdot((c+7)v-c+7)\cdot((c-7)v-c-7),$$
$$\frac{m(u', v')}{m(u, v)} = 28\frac{N_{uv}}{D_{uv}}\cdot J[K_3](u, v).$$

At this point, the mapping K_3 does not seem to have a preserved meromorphic 2-form.

However, if a meromorphic 2-form exists, we know [10] that the fixed points of the mapping for which the Jacobian is $J \neq 1$ should be located on a covariant curve corresponding to the meromorphic 2-form. For this mapping, there are four fixed points of order 1 where $J = 1$ and two fixed points of order 1 where $J \neq 1$. The latter reads as $(u = -5/2 \pm \sqrt{21}/2, v = 1)$ and is neither on $(u-1)(v-u) = 0$ nor on $C_2 = 0$. The line $v - 1 = 0$ should be covariant as this appears clearly from expression (15) of the birational mapping K_3.

Producing this line $v - 1 = 0$ from the iterates of V_1 may call for a tricky analysis. Instead, and since this is equivalent, we consider the backward mapping and its 'long' post-critical set. Canceling the Jacobian (or its inverse) of K_3^{-1}, one obtains (among others) the algebraic curve

$$28u^2 + 56(1+v)u - (c^2+35)v^2 + 2(c^2-35)v - c^2 - 35 = 0.$$

Eliminating the variable u between this curve and the iterates $K_3^{-n}(u, v)$ will give the line $v - 1 = 0$ common to both components.

Combining the covariant curves $(u-1)(v-u) = 0$, $C_2 = 0$ and the new line $v - 1 = 0$, one obtains

$$m_3(u, v) = \frac{(u-1)(v-u)\cdot}{v-1} \cdot ((c+7)v - c + 7)((c-7)v - c - 7),$$
$$\frac{m_3(u', v')}{m_3(u, v)} = J[K_3](u, v),$$

giving the corresponding meromorphic 2-form $\mathrm{d}u\mathrm{d}v/m_3(u, v)$.

Appendix B. Another birational mapping: K_5

We consider the collineation C corresponding to matrix (A.1) but with the mapping constructed as $K_5 = C \cdot j$. This mapping arises in [51] and was considered in [2, 52]. For the values of the parameters $a = b = -1 + q^2$, $c = d = q$, it was shown [51] that it has an algebraic invariant for all values of the parameter q. Here, we take the parameters as $a = b = q^2$, $c = d = q$ and the birational mapping is non-integrable for generic values of q. The birational mapping reads as

$$K_5: \quad (u, v) \longrightarrow (u', v') = \left(\frac{q^2 \cdot (1+v)u + 2v}{D_{uv}}, 1 - \frac{2q \cdot (v-1)u}{D_{uv}} \right), \qquad \text{(B.1)}$$

$$D_{uv} = (q-1) \cdot uv - (q+1) \cdot u + 2v,$$

with the Jacobian

$$J[K_5](u, v) = -8(1+q^2) \cdot q \cdot \frac{uv}{D_{uv}^3}. \qquad \text{(B.2)}$$

Its critical set reads as

$$\mathcal{E}(K_5) = \{V_1, V_2, V_3\} = \{(v = 0); (D_{uv} = 0); (u = 0)\}, \qquad \text{(B.3)}$$

and the first iterates of the critical set are given by

$$K_5^n(V_1) = \left(-\frac{q^2}{q+1}, \frac{1-q}{q+1} \right) \longrightarrow \left(\frac{1-q^2-q^4}{1+q^4}, \frac{1-q^4}{1+q^4} \right),$$
$$K_5^n(V_2) = (\infty, \infty) \rightarrow \left(\frac{q^2}{q-1}, \frac{1+q}{1-q} \right) \rightarrow \left(\frac{1-q^2-q^4}{1-q^4}, \frac{1+q^4}{1-q^4} \right).$$

Those of $K_5^n(V_3)$ are the same as $K_5^n(V_2)$ after blowing down first on point $(1, 1)$.

The post-critical set for the *backward* mapping is also 'long' and the orbits have similar expressions. The post-critical set, for both forward and backward mapping, is 'long'. The degree growth in the parameter q of the *iterates of the critical set* being exponential, there is no preserved meromorphic 2-form. The phase portraits of this mapping show a foliation in the plane with an infinity of leaves, similar to the mapping analyzed in [10]. Note that the line $v - 1 = 0$ is covariant as easily seen from the expression of K_5. The phase portrait however shows no accumulation of points near this line.

Appendix C. A mapping by Bedford and Kim [40]

Consider the following birational mapping taken from Bedford and Kim [40]:

$$K_{\rm BK}: \quad (u, v) \longrightarrow (u', v') = \left(v, \frac{v+a}{u+b}\right), \tag{C.1}$$

which is non-integrable for generic a and b, the degree growth of the iterates being $\lambda = 1.3247\ldots$ given by $1 - t^2 - t^3 = 0$. Its Jacobian reads as

$$J[K_{\rm BK}](u, v) = \frac{v+a}{(u+b)^2}. \tag{C.2}$$

The critical set is given by

$$\mathcal{E}(K_{\rm BK}) = \{(v = -a); (u = -b)\}, \tag{C.3}$$

where only $v = -a$ blows down on points. The post-critical set is given by

$$(u, -a) \to (-a, 0) \to \left(0, \frac{a}{b-a}\right) \to \left(\frac{a}{b-a}, \frac{a(1-a+b)}{b(b-a)}\right) \to \cdots.$$

The degree growth in the iterates of the parameters a and b is $\lambda = 1.3247\ldots$, i.e. the post-critical set is *non-integrable*. Our claim is that there are no algebraic covariant curves for *generic* a and b for this birational mapping.

However, the parameters a and b can be such that the degree growth of the post-critical set is reduced to $\lambda = 1$, i.e. the post-critical set becomes integrable. In this case, algebraic covariant curves can exist.

Bedford and Kim have shown [40] three cases of *reduction of complexity in the post-critical set*. They read as

$$a = -\frac{\mu(-1+\mu^2+\mu^3)}{(\mu+1)^2}, \qquad b = \frac{1-\mu^5}{\mu^2+\mu^3}, \tag{C.4}$$

$$a = \frac{\mu(1+\mu+\mu^2)}{(\mu+1)^2}, \qquad b = \frac{(\mu-1)(1+\mu+\mu^2)}{\mu(\mu+1)}, \tag{C.5}$$

$$a = 1+\mu, \qquad b = \mu - \frac{1}{\mu}, \tag{C.6}$$

and the algebraic covariant curves (see [40] for details) read as $m(u, v) = 0$, respectively, with

$$\begin{aligned} m(u, v) = {} & (-1+v)\mu^9 - (1+u-2v)\mu^8 - 2v(u-1)\mu^7 + (-4uv + u^2 + 1 - v^2 + 2u \\ & + 2v)\mu^6 + (2u^2 - 4uv + u^2 v + 2u - v^2)\mu^5 + (-4uv + 2u^2 v - 2v + uv^2 \\ & + u^2)\mu^4 + v(u^2 - 2 + 2uv - 4u + v)\mu^3 + v(uv + 2v - 2u)\mu^2 + 2v^2\mu + v^2, \end{aligned}$$

$$m(u,v) = \mu^6 + (1+v+2u)\mu^5 + (3u+u^2+2v+2uv+1)\mu^4$$
$$+ (2u^2+u^2v+2u+4uv+v^2+2v)\mu^3 + (v^2+4uv+uv^2+u^2+2u^2v+2v)\mu^2$$
$$+ v(2u+v+2uv+u^2)\mu + v^2(1+u),$$

and

$$m(u,v) = \mu^4 + (v+2u+1)\mu^3 + (1+u)(u+2v)\mu^2 + v(2u+u^2+v)v\mu + uv^2.$$

All these covariant expressions yield 2-forms preserved up to a factor

$$\frac{m(u',v')}{m(u,v)} = \mu \cdot J[K_{BK}](u,v). \qquad \text{(C.7)}$$

Note that for the three cases (C.4)–(C.6), while the post-critical set is 'integrable', allowing the occurrence of algebraic covariant curves, the mapping K_{BK} *itself* is still with degree growth $\lambda = 1.3247\ldots$.

Appendix D. A parameter-free birational mapping

This mapping is taken from [35] and originates from lattice statistical mechanics and is related to mapping K_1. It is parameter-free, non-integrable and reads as

$$K_6: \quad (u,v) \longrightarrow (u',v') = \left(v, \frac{1+v-uv}{uv}\right). \qquad \text{(D.1)}$$

Its Jacobian reads as

$$J[K_6] = \frac{1+v}{u^2 v}. \qquad \text{(D.2)}$$

The orbits of the critical set read as

$$K_6^n(v=-1) = (-1,-1) \longrightarrow (-1,-1),$$
$$K_6^n(u=0) = (v,\infty) \rightarrow (\infty,(1-v)/v) \rightarrow ((1-v)/v,-1) \rightarrow (-1,-1),$$
$$K_6^n(v=0) = (0,\infty) \quad \rightarrow \quad (\infty,\infty) \quad \rightarrow \quad (\infty,-1) \quad \rightarrow \quad (-1,-1).$$

The post-critical set is 'short' and there is an attracting set which is the point $(-1,-1)$.

For the backward mapping, the orbits of the critical set are

$$K_6^{-n}(u=-1) = (0,-1) \quad \rightarrow \quad (\infty,0) \quad \rightarrow \quad (1,\infty) \quad \rightarrow \quad (0,1),$$
$$K_6^{-n}(u=0) = (\infty,0) \quad \rightarrow \quad (1,\infty) \quad \rightarrow \quad (0,1) \quad \rightarrow \quad (\infty,0),$$
$$K_6^{-n}(v=-1) = (\infty,u) \quad \rightarrow \quad (1/(1+u),\infty) \quad \rightarrow \quad (0,1/(1+u))$$
$$\longrightarrow (\infty,0) \longrightarrow (1,\infty) \quad \longrightarrow \quad (0,1).$$

The post-critical set is short and there is an attracting set which is the cycle of order 3: $(\infty,0) \rightarrow (1,\infty) \rightarrow (0,1)$.

Note that one may remark (see the form of the mapping) that $v+1=0$ is covariant for the forward mapping, but it is not covariant for the backward mapping, where it gives birth to an attracting point of order 3.

The Jacobian evaluated at the successive fixed points gives the following. There is one fixed point of order 1 with[19] $J=2$. There is no fixed point of order 3. There is only one fixed point for the orders 2, 4, 5, 6 with respectively $J = 2, 2, -5, 1$.

[19] For the backward mapping, the value of the Jacobian evaluated at the fixed point is the inverse of the value corresponding to the forward mapping.

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